Chapter 11
HYDROFORMING

11.1. INTRODUCTION.

In hydroforming, or fluid pressure forming, sheet is formed against a die by fluid pressure. In many cases, a flexible diaphragm is placed on the sheet and it is then formed into a female die cavity as shown in Figure 11.1. The advantage of the process is that die construction is simpler and the process may be economical for making smaller numbers of parts. A disadvantage is that very high pressures may be required and the cycle time is greater than for stamping in a mechanical press.

Figure 11.1. (a) A typical sheet metal part. (b) The arrangement for pressure, or hydroforming into a female die.

Hydroforming is also used to form tubular parts such as brackets for bicycle frames or pipe fittings, as shown in Figure 11.2. Axial force may be applied to the tube as well as internal pressure; this creates compressive stress in one direction so that elements of the tube deform without thinning and tearing is delayed. With specially designed forming machines, a large number of parts can be formed by this process at low cost.
Another application is forming tubular parts such as vehicle frame components. A round tube is bent and then placed in a die as shown in Figure 11.3. It is then pressurized internally and formed to a square section.

In this chapter we do not attempt the analysis of a complete process, but break the process down into elements of simple geometry and model each of these separately. This will illustrate the limits of the process and show the influence of friction, geometry and
material properties. The free expansion of a tube is first studied. Forming a square section from a round tube in the so-called high-pressure process is analysed and then it is shown how some of the process limits in this process can be overcome in a sequential forming process.

11.2. FREE EXPANSION OF A CYLINDER BY INTERNAL PRESSURE.

Expansion of a round tube without change in length is analysed. The tube will deform in plane strain, i.e., the strain in the axial direction will be zero. Initially the tube will remain circular and the radius will increase. The expansion of a cylindrical element in this mode is illustrated in Figure 11.4. The strain and stress states, for an isotropic material, are:

\[ \varepsilon_{\theta} ; \quad \varepsilon_{\phi} = \beta \varepsilon_{\theta} = 0 ; \quad \varepsilon_{r} = -(1 + \beta) \varepsilon_{\theta} = -\varepsilon_{\theta} ; \]
\[ \sigma_{\theta} ; \quad \sigma_{\phi} = \alpha \sigma_{\theta} = \frac{1}{2} \sigma_{\theta} ; \quad \sigma_{r} = 0. \]  

(11.1)

ie., \( \beta = 0, \alpha = \frac{1}{2} \).

Figure 11.4. Element of a circular tube with internal pressure.

If the material properties obey the stress strain law,

\[ \bar{\sigma} = K(\bar{\varepsilon})^{n}, \]

When the material is deforming we obtain from Equations 2.18 and 2.19,

\[ \sigma_{\theta} = \frac{2}{\sqrt{3}} \bar{\sigma} = \frac{2}{\sqrt{3}} \sigma_{f} ; \quad \text{and} \quad \bar{\varepsilon} = \frac{2}{\sqrt{3}} \varepsilon_{i} \]  

(11.2)
From Section 7.2, the principal radii at any region of the tube are,
\[ \rho_2 = \infty; \quad \text{and} \quad \rho_1 = r \quad (11.3) \]

From Equation 7.3, the hoop tension is,
\[ T_\theta = \sigma_\theta t = p.r \quad (11.4) \]

and as this is plane strain, \( T_\theta = \alpha.T_\theta = \frac{1}{2}T_\theta \).

For the tube to yield, the pressure, from Equations 11.2 and 11.4, is,
\[ p = 2 \frac{\overline{T}}{\sqrt{3}} = 2 \frac{\sigma_f.t}{\sqrt{3}r} \quad (11.5) \]

If the tube is initially of thickness, \( t_0 \), and radius, \( r_0 \), the current hoop strain and thickness are given by,
\[ \varepsilon_\theta = \ln \frac{r}{r_0} \quad \text{and} \quad t = t_0 \cdot \exp \varepsilon_\theta = t_0 \cdot \exp(-\varepsilon_\theta) = \frac{t_0.r_0}{r} \quad (11.6) \]

From this, we obtain the pressure characteristic for expansion as,
\[ p = 2 \frac{\overline{\sigma}}{\sqrt{3}}.K \left( \frac{2}{\sqrt{3}}.\ln \frac{r}{r_0} \right)^n t_0 \cdot \frac{r_0}{r^2} \quad (11.7) \]

From the form of Equation 11.7, we see that for a strain hardening material, \( n>0 \), the pressure will tend to increase as the material deforms. On the other hand, if the tubular element is allowed to expand freely, the tube wall will thin and the radius increase; both effects will tend to decrease the pressure. At some point, the pressure will reach a maximum as the opposing effects balance.

We need also to consider the possibility of the tube wall necking and splitting. As this is plane strain deformation, the loading path will be along the vertical axis in the strain space, Figure 5.16. Splitting would be expected approximately when the hoop strain has approximately the value, \( n \). Thus the limiting case is when,
\[ \ln \frac{r^*}{r_o} = n \quad \text{and} \quad p^* = \frac{2}{\sqrt{3}}.K \left( \frac{2}{\sqrt{3}.n} \right)^n \frac{t_0}{r_0} \exp(-2n) \quad (11.8) \]

where, \( r^* \), is the radius at which the tube becomes unstable and will neck.
11.3. FORMING A CYLINDER TO A SQUARE SECTION.

A common operation is forming a round tube into a square die as illustrated in Figure 11.3; in the middle of the part, the tube will deform in plane strain. Various stages of the process are shown in Figure 11.5.

![Diagram of forming a cylinder to a square section](image)

In Figure 11.5(c), the tube has been partially expanded so that the wall is touching the die up to the point, $A$. The contact length is, $\ell$, and assuming that the thickness is small compared with the radius, the current contact length is, $\ell = r_0 - r$. During an increment in the process, the contact length increases to, $\ell + d\ell$, and the corner radius decreases to $r + dr$, where, $dr$, is a negative quantity, i.e., $dr = -d\ell$.

In the contact zone, the tube will be pressed against the die wall by the internal pressure. If the material slides along the die, friction will oppose the motion and the tension will change. At some point, the tension will be insufficient to stretch the wall and there will be a sticking zone as shown in Figure 11.6.
Figure 11.6. (a) Part of the tube wall in contact with the die during forming of a cylinder to a square section. (b) Element of the tube wall in the sliding zone.

The equilibrium equation for the element in Figure 11.6(b) is,

$$T_\theta + dT_\theta = T_\theta + \mu p ds.1$$

or,

$$\frac{dT_\theta}{ds} = \mu p$$

(11.9)

The deformation process is stable as long as the tension increases with strain so we assume here that the tension in the unsupported corner will continue to increase as the radius becomes smaller. Referring to Section 3.6, the slope of the tension versus strain curve for plane strain is positive for strains of, \( \varepsilon_\theta \leq n \). Thus in Figure 11.6, the tension to yield the tube will increase as the tube wall thins. For the tube in contact with the die, the greatest tension will be at the tangent point. Equation 11.9 shows that due to friction, the tension decreases linearly towards the centre-line; the distribution is shown in Figure 11.7. To the right of the point where sliding ceases, the tension in the tube wall is less than that required for yielding and there is no further deformation in this sticking region. The critical point is where the thickness is, \( t_s \). For a material obeying the stress strain law,

$$\bar{\sigma} = K(\bar{\varepsilon})^n$$

from Equations 11.2, the hoop stress is given by,
\[
\sigma_\theta = \frac{2}{\sqrt{3}} K \left( \frac{2}{\sqrt{3}} \varepsilon_\theta \right)^n 
\]

As this is a plane strain process,

\[
\varepsilon_\theta = -\varepsilon_t = \ell n \frac{t_0}{t}
\]

The tension at the critical point in Figure 11.7 separating sliding and sticking is,

\[
T_{\theta_c} = \sigma_\theta t_s = \frac{2}{\sqrt{3}} K \left( \frac{2}{\sqrt{3}} \ell n \frac{t_0}{t_s} \right)^n t_s \quad (11.10)
\]

In the sticking zone, there is no sliding and the slope of the tension curve in Figure 11.7 will be less than, \( \mu p \).

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![Diagram](attachment:image.png)

**Figure 11.7.** Distribution of tension in that part of the tube in contact with the die wall during forming of a round tube to a square section.

The distribution of thickness in the wall can be determined by an incremental analysis. In this work, the extreme cases, either with no friction at the die wall or with sticking friction along the entire contact length will be considered.
11.3.1. Tube Forming in a Frictionless Die.

If contact between the tube and the die is frictionless, at any instant, the tension and also the thickness at any point around the tube will be uniform. The current perimeter length of one-quarter of the tube in Figure 11.5 is,

$$\frac{\pi}{2} r + 2\ell = \frac{\pi}{2} r + 2(r_0 - r) = 2r_0 - r\left(2 - \frac{\pi}{2}\right)$$  \hspace{1cm} (11.11)

As there is no change in volume of the tube material,

$$t \left[2r_0 - r\left(2 - \frac{\pi}{2}\right)\right] = t_0 \frac{\pi}{2} r_0$$

or,

$$t = \frac{t_0}{\frac{4}{\pi} - r\left(\frac{4}{\pi} - 1\right)}$$  \hspace{1cm} (11.12)

As this is a plane strain process, the hoop strain is,

$$\varepsilon_\theta = -\varepsilon_t = \ell n t_0$$

and for a material obeying the stress, strain law, $\bar{\sigma} = K\bar{\varepsilon}^n$, the hoop stress is

$$\sigma_\theta = \frac{2}{\sqrt{3}} K \left(\frac{2}{\sqrt{3}} \ell n \frac{t_0}{t}\right)^n$$  \hspace{1cm} (11.13)

The pressure required to deform the unsupported corner of radius, $r$, can be determined from Equation 11.4.

11.3.2. Tube Forming with Sticking Friction (or Very High Friction).

If the tube sticks to the die wall as soon as it touches it, then for a circular tube that initially just fits inside the die as shown in Figure 11.5(a), the thickness at the first point of contact will be, $t_0$. As the tube becomes progressively attached to the wall, the thickness at the tangent point this will decrease so that at A, in Figure 11.5(c), it has the value, $t$. For a unit length perpendicular to the plane of the diagram, the volume of material in the arc, $AA$, is, $\pi r t / 2$. This volume will remain unchanged during the increment, and as this is plane strain, equating the volumes before and after the increment, we obtain,
\[
\frac{\pi}{2} \left[(r + dr) + 2d\ell\right]t + dt = \frac{\pi}{2} rt
\]

From above, \( \ell = r_0 - r \), and, \( d\ell = -dr \), hence,

\[
\frac{dt}{t} = \left(\frac{4}{\pi} - 1\right) \frac{dr}{r}
\]

Integrating with the initial conditions of, \( t = t_0 \), at, \( r = r_0 \), we obtain,

\[
t = t_0 \left(\frac{r}{r_0}\right)^{\frac{4}{\pi} - 1}
\]

The hoop strain and pressure to continue the deformation can then be determined following the same approach as in Section 11.3.1. The wall thickness will be non-uniform varying from the initial thickness at the center-line of the die to a minimum at the unsupported corner radius. For a given corner radius, the corner thickness will be less than for the frictionless case and failure could occur earlier in the process.

11.3.3. Failure in Forming a Square Section.

The above analysis neglects the effect of unbending under tension at the tangent point, \( A \), in Figure 11.5. This will be similar to the effects described in Section 10.5, and cause additional thinning and work-hardening of the tube wall. In plane strain forming of a tube of uniform section, the process is possible provided that the tension in the deforming wall continues to increase with forming. If it reaches a maximum, necking and failure of the tube will take place. As indicated in Section 5.4, for plane strain, the limit for a power law hardening material is approximately when, \( n = \bar{\epsilon} \).

In forming the corner radius, the pressure required will increase as the radius decreases, as shown by Equation 11.5. Even though very high pressure equipment may be used, the limitation on corner radius is usually significant and this, together with the forming limit strain, are the first things that should be calculated in preliminary design.

11.4. CONSTANT THICKNESS FORMING.

In plane strain expansion, as indicated above, the material will split approximately when the hoop strain reaches the value of the strain-hardening index, \( n \). The strain path is illustrated in Figure 11.8(a). To obtain the required strain in the part, high strain-hardening material is used, but this exacerbates the problem of high pressure for forming.
Large strains are potentially possible in processes such as those illustrated in Figure 11.2, and it would be advantageous to choose a low strain-hardening material. To avoid splitting, a constant thickness strain path, $\beta = -1$, would be the objective and this strain path together with the forming limit for a low $n$ material is illustrated in Figure 11.8(b).

The design of processes that achieve constant thickness deformation is not easy, but we consider some cases of simple geometry.

11.4.1. Constant Thickness Deformation for a Tube Expanded by Internal Pressure.

In a constant thickness process, the hoop and axial tensions and stresses would be equal and opposite. The hoop tension, from Equation 11.4, is, $T_\theta = pr$; the axial tension or traction, which is compressive, is, $T_\phi = -pr$. To achieve this, an axial force must be applied to the tube as illustrated in Figure 11.9; the axial force is,

$$ F = -2\pi r T_\phi = 2\pi r^2 p $$

(11.16)

From Equations 2.18 and 2.19, the effective strain and stress are,

$$ \varepsilon = \frac{2}{\sqrt{3}} \varepsilon_\theta = \frac{2}{\sqrt{3}} \frac{\ell n r}{r_0} \quad \text{and} \quad \sigma_\theta = \frac{1}{\sqrt{3}} \tilde{\sigma} $$

(11.17)

where, $\sigma_\theta$, is given by Equation 11.4. For a given expansion in a material obeying the stress strain law, $\tilde{\sigma} = K(\varepsilon)^n$, and noting that the thickness remains constant, we obtain the pressure required to continue the process as,

$$ p = \frac{\sigma_\theta t}{r} = \frac{1}{\sqrt{3}} K \left( \frac{2}{\sqrt{3}} \frac{\ell n r}{r_0} \right)^n \frac{t_0}{r} $$

(11.18)

Comparing this with Equation 11.7, we see that the pressure is reduced for constant thickness expansion, compared with plane strain, but of course an axial force is required and this will do work as well as the pressure.
11.4.2. Effect of Friction on Axial Compression.

To achieve axial compression on an element, a force is applied to the end of the tube as shown in Figure 11.2. The effect of this force is local because friction between the tube and the die will cause it to diminish with distance from the point of application of the force. The case for a simple tube is shown in Figure 11.10.
In this diagram, a plunger on the left applies compression to a tube. At some distance, \( z \), the tension on one side of an element of width, \( dz \), is, \( T_\phi \), and on the other side, \( T_\phi + dT_\phi \). Around this elemental ring there is a contact pressure, \( q \), and a friction stress, \( \mu q \). The equilibrium equation for the element is,

\[
2\pi r(T_\phi + dT_\phi) = 2\pi r T_\phi + 2\pi r \mu q dz
\]

or,

\[
dT_\phi = \mu q dz \quad (11.19)
\]

This shows that the tension, or traction increases, i.e., becomes more tensile as, \( z \), increases. It starts as most compressive at the plunger and the compression decreases linearly with distance from the end of the tube. For this reason it is not possible to obtain axial compression in the middle of parts such as shown in Figure 11.3, but in shorter parts, such as Figure 11.2, axial compression can be effective in preventing thinning and necking. In parts of complicated shape, such as in Figure 11.2, it is found that to ensure that thinning does not occur in any critical regions, some places will become thicker, but this is usually acceptable.

11.4. **LOW-PRESSURE OR SEQUENTIAL HYDROFORMING.**

In forming parts such as in Figure 11.3, a technique has evolved that avoids the use of very high pressures. To form a square section, an oval tube is compressed during closure of the die and the internal fluid only serves to keep the tube against the die as shown in Figure 11.11.

![Figure 11.11](image-url)

Figure 11.11. Forming (a) an oval tube into (b) a square or rectangular section in a low-pressure hydroforming process.
In this process, the periphery of the tube remains approximately constant and tearing is not likely to be a limiting factor. The mechanics of the process can be illustrated by considering the case of a circular arc of a tube being formed into a corner as shown in Figure 11.12. In this case the process is symmetric about the diagonal; this is not quite the same as that shown in Figure 11.11, but will illustrate the principle.

Figure 11.12. Deformation of the corner of a tube in a low-pressure hydroforming process. (a) The forces acting, and (b), the bending moment diagram on the arc, AB.

In forming the tube into the corner, the wall is bent to a sharper radius at \( B \), and straightened at \( A \). The bending moment diagram is shown in Figure 11.12(b). The stress distribution at \( B \), is shown in Figure 11.13, for a rigid, perfectly plastic material.

Figure 11.13. Stress distribution at the corner of a tube being deformed as in Figure 11.12.
As shown in Section 10.4, the bending moment required to deform the tube at, \( B \), is,

\[
M_B = \frac{S_t^2}{4} \left[ 1 - \left( \frac{T_{yB}}{T_y} \right)^2 \right] = M_p \left[ 1 - \left( \frac{T_{yB}}{T_y} \right)^2 \right]
\]  

(11.20)

where, \( S_t \), is the plane strain flow stress, \( T_y \), the yield tension, and, \( M_p \), the moment to bend the wall in the absence of tension or compression i.e. the fully plastic moment determined in Section 6.5.2. Thus by applying compression to the tube wall, the tube can be bent easily at the corner and the tangent point; a plastic hinge will form at, \( B \), to bend the tube, and one at, \( A \), to straighten it. In this simple analysis, bending will only occur at, \( A \), and, \( B \), and the region between will remain unchanged. This is not very realistic as material is never completely rigid, perfectly plastic, but it is observed that in this process, the radius of curvature in the unsupported corner is not constant; it is least at the point, \( B \).

In performing this operation, the tube is filled with fluid, usually water, and sealed as it is placed in the die as in Figure 11.11. As the internal volume diminishes in forming the oval to the square section, fluid is expelled from the tube. The pressure is regulated using a pressure control valve and is maintained at a sufficient level to keep the tube against the die walls and prevent wrinkling. Once the die is closed, the pressure may be increased to improve the shape. It may be seen that the overall perimeter of the tube does not change very much during forming and therefore splitting is avoided.

11.5. SUMMARY.

In this chapter, several example of forming using fluid pressure have been examined. There are many other applications of these principle and the following factors should be borne in mind in considering fluid forming.

- Very high fluid pressures are required to form small fillet radii.
- Forming equipment becomes expensive as the pressures become high.
- Very high strains can be achieved if compressive forces are applied to the part as well as pressure and approximately constant thickness deformation obtained.
- Forming pressures can be reduced if controlled buckling under compressive forces is obtained.
Exercises

Ex. 11.1. A circular tube with a radius $R$ and thickness $t_o$ is deformed into a die with a square cross section through high pressure hydroforming. Find the relation between the corner radius and the internal pressure in the two cases:

a) There is no friction at the die metal interface.

b) The tube material fully sticks to the die surface.

The material obeys effective stress-strain law: $\bar{\sigma} = K(\epsilon_o + \bar{\epsilon})^n$.

Ex. 11.2. A mild steel tube of 180 mm diameter and thickness of 4 mm is to be expanded by internal pressure into a square section. The maximum pressure available is 64MPa. The material has a stress strain curve of $\bar{\sigma} = 700(0.002 + \bar{\epsilon})^{0.2}$

Determine the minimum corner radius that can be achieved.

Ex. 11.3. In a tube hydroforming process, a square section was formed from a circular tube. The initial tube had a radius of $R$. The current corner radius is $r$. The maximum thickness along the tube was $t_1$. The corner thickness was $t_2$. The material strain hardens according to the relation $\bar{\sigma} = K(\epsilon_o + \bar{\epsilon})^n$.

Assuming that the thickness varies linearly along the tube wall, calculate the average friction coefficient between the tube wall and die surface.

Ex. 11.4. Find the pressure required to expand a circular tube from initial radius $r_o$ to a final radius $r$. The material hardens as $\sigma = K\epsilon^n$, for two cases

a) Tube ends are free to move axially.

b) Tube ends are restricted from axial movements.

Calculate also the pressure at instability.

Ex. 11.5. A thin walled torus with a radius $R$, and cross-sectional radius $r$, and a thickness $t$, is subjected to internal pressure. Find the stress distribution in the torus and the internal pressure at first yield and that corresponding to full yielding of the torus.
Chapter 11 SOLUTIONS

Ex. 11.1

a) For the frictionless case:

The current radius of the tube at the corner is \( r \). The tube wall thickness will be uniform.

From volume constancy

\[
t (2 (R - r) + \frac{\pi}{2} r) = \frac{\pi}{2} R t_o
\]

re-arranging

\[
t = \frac{4}{\pi - \frac{4 - \pi}{\pi}} \frac{t_o}{R}
\]

For the plane strain condition

\[
\varepsilon = \frac{2}{\sqrt{3}} \varepsilon_o = \frac{2}{\sqrt{3}} \ln\left(\frac{t_o}{t}\right) \quad \text{and} \quad \sigma = \frac{\sqrt{3}}{2} \sigma_i
\]

\[
\sigma_o = \frac{2}{\sqrt{3}} K \left[ \varepsilon_o + \frac{2}{\sqrt{3}} \ln\left(\frac{4}{\pi} - \frac{4 - \pi}{\pi} \frac{r}{R}\right) \right]^{n}
\]

Internal pressure is given by:

\[
p = \left(\frac{\sigma_o t}{r}\right) = \frac{2}{\sqrt{3}} \frac{t_o K}{R} \left[ \varepsilon_o + \frac{2}{\sqrt{3}} \ln\left(\frac{4}{\pi} - \frac{4 - \pi}{\pi} \frac{r}{R}\right) \right]^{n}.
\]

b) For the full sticking case:

The relation between the corner radius and the corner thickness can be obtained from the incremental relation

\[
2 t \, dr + \frac{\pi}{2} (r - dr)(t - dt) = \frac{\pi}{2} r t
\]

\[
\int_r^t (\frac{4}{\pi} - 1) \frac{dr}{r} = \int_t^{t_o} \frac{dt}{t}
\]

\[
t = t_o \left(\frac{r}{R}\right)^{\frac{4}{\pi - 1}}
\]

Circumferential stress
\[ \sigma_i = \frac{2}{\sqrt{3}} K \left[ \varepsilon_o + \frac{2}{\sqrt{3}} \left( \frac{4}{\pi} - 1 \right) \ln \left( \frac{R}{r} \right) \right]^n \]

The internal pressure is given by:

\[ p = \frac{\sigma_i t}{r} = \frac{2 t_o}{\sqrt{3} r} \left( \frac{4}{\pi} - 1 \right) K \left[ \varepsilon_o + \frac{2}{\sqrt{3}} \left( \frac{4}{\pi} - 1 \right) \ln \left( \frac{R}{r} \right) \right]^n \]

**Ex. 11.2**

The minimum corner radius is limited either by necking or by the maximum fluid pressure available.

Since \[ t = \frac{t_o}{\frac{4}{\pi} - \left( \frac{4}{\pi} \right) \frac{r}{R}} \],

\[ \varepsilon_o = -\varepsilon_i = \ln \frac{t_o}{t} , \quad \varepsilon_o^* = n = 0.2 \]

\[ t = 3.27 , \quad r = 17 mm. \]

The internal pressure required to form corner radius \( r \) is given in Ex. 11.1. Substituting the material properties and die geometry into the equation, we obtain:

<table>
<thead>
<tr>
<th>( r (mm) )</th>
<th>( t (mm) )</th>
<th>( \sigma_o (MPa) )</th>
<th>( p (MPa) )</th>
</tr>
</thead>
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<td>38.00</td>
<td>3.45</td>
<td>566.85</td>
<td>51.52</td>
</tr>
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<td>37.00</td>
<td>3.44</td>
<td>568.86</td>
<td>52.96</td>
</tr>
<tr>
<td>36.00</td>
<td>3.44</td>
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<td>54.47</td>
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<td>3.43</td>
<td>572.78</td>
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<td>574.70</td>
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</tr>
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<td>32.00</td>
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<td>578.44</td>
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<tr>
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<td><strong>63.48</strong></td>
</tr>
<tr>
<td>30.00</td>
<td>3.38</td>
<td>582.07</td>
<td>65.63</td>
</tr>
<tr>
<td>29.00</td>
<td>3.37</td>
<td>583.84</td>
<td>67.92</td>
</tr>
</tbody>
</table>

Therefore, the minimum corner radius is 31 mm, and is limited by the maximum pressure available.
Ex. 11. 3

In order to calculate the initial tube thickness, we use volume constancy.

\[
2 \left( \frac{t_1 + t_2}{2} \right) L + \frac{\pi}{2} r t_2 = \frac{\pi}{2} R t_o
\]

where \( L = (R - r) \)

\[
t_o = \frac{1}{R} \left[ 2 \left( \frac{t_1 + t_2}{\pi} \right) (R - r) + rt_2 \right]
\]

The forces that existed at each end of the wall can be calculated as:

\[
F_1 = \sigma_1 t_1 = \frac{2}{\sqrt{3}} K [\varepsilon_o + \frac{2}{\sqrt{3}} \ln \left( \frac{t_o}{t_1} \right)]^n t_1
\]

\[
F_2 = \sigma_2 t_2 = \frac{2}{\sqrt{3}} K [\varepsilon_o + \frac{2}{\sqrt{3}} \ln \left( \frac{t_o}{t_2} \right)]^n t_2
\]

The final internal pressure at the current forming radius can be found from:

\[
F_2 = pr
\]

thus \( p = \frac{F_2}{r} \)

To calculate the average friction coefficient from force balance:

\[
\mu_{av} p L = F_2 - F_1
\]

\[
\mu_{av} = \frac{F_2 - F_1}{pL}
\]
Ex. 11.4

a) For the free ends, there is no stress in the axial direction, $\sigma_2 = 0$.

For thin cylinder $\sigma_3 = 0$.

$$\varepsilon_1 = \ln\left(\frac{L}{r_0}\right)$$

$$\varepsilon_2 = \varepsilon_3 = -\frac{1}{2} \varepsilon_1$$

$$\ln\left(\frac{t}{t_o}\right) = -\frac{1}{2} \ln\left(\frac{r}{r_o}\right)$$

$$\frac{t}{t_o} = \sqrt[3]{\frac{r_o}{r}}$$

Effective stress and strains are:

$$\bar{\sigma} = \sigma_1 \quad \text{and} \quad \bar{\varepsilon} = \varepsilon_1$$

The internal pressure is calculated as

$$p = \frac{\sigma_1 t}{r}$$

$$p = K t_o \sqrt[3]{\frac{r_o}{r^{3/2}}} [\ln\left(\frac{r}{r_o}\right)]^n$$

To calculate the expanded radius at instability, let $dp = 0$.

This leads to:

$$\ln\left(\frac{r}{r_o}\right) = \frac{2n}{3} \quad \text{or} \quad r = r_o e^{2n/3}$$

b) For restricted ends, there is no strain in the axial direction, $\varepsilon_2 = 0$.

$$\varepsilon_1 = \ln\left(\frac{L}{r_0}\right)$$

$$\varepsilon_3 = -\varepsilon_1$$

$$\ln\left(\frac{t}{t_o}\right) = -\ln\left(\frac{r}{r_o}\right)$$

$$\frac{t}{t_o} = \frac{r_o}{r}$$

Effective stress and strains are:

$$\bar{\sigma} = \frac{\sqrt{3}}{2} \sigma_1 \quad \text{and} \quad \bar{\varepsilon} = \frac{2}{\sqrt{3}} \varepsilon_1$$
\[ \sigma_1 = \frac{2}{\sqrt{3}} K \left( \frac{2}{\sqrt{3}} \ln \left( \frac{r}{r_o} \right) \right)^n \]

The internal pressure is calculated as

\[ p = \frac{\sigma_1 t}{r} = \frac{\sigma_1 t}{r} \]

\[ p = \left( \frac{2}{\sqrt{3}} \right)^{n+1} K \frac{t_o r_o}{r^2} [\ln \left( \frac{r}{r_o} \right)]^n \]

To calculate the expanded radius at instability, let \( dp = 0 \).

This leads to:

\[ \ln \left( \frac{r}{r_o} \right) = \frac{n}{2} \quad \text{or} \quad r = r_o e^{n/2} \]
Ex. 11.5

Solution:

\[
\frac{\sigma_1 + \sigma_2}{\rho_1 + \rho_2} = \frac{p}{t}
\]

\[
\frac{\sigma_1}{r} - \frac{\sigma_2}{R \sec \psi - r} = \frac{p}{t}
\]

Force balance in the z-direction (the direction of the axis of the torus):

\[
\pi \left( R^2 - (R - r \cos \psi)^2 \right) p = 2\pi (R - r \cos \psi) \ t \sigma_1 \cos \psi
\]

rearranging:

\[
\sigma_1 = \frac{pr}{2t} \left( \frac{2R - r \cos \psi}{R - r \cos \psi} \right)
\]

substitute to get \( \sigma_2 \):

\[
\sigma_2 = \left( \frac{\sigma_1}{r} - \frac{p}{t} \right) (R \sec \psi - r)
\]

\[
\sigma_2 = \left[ \frac{p}{2t} \left( \frac{2R - r \cos \psi}{R - r \cos \psi} \right) - \frac{p}{t} \right] \frac{R - r \cos \psi}{\cos \psi}
\]

\[
\sigma_2 = \frac{pr}{2t}
\]

which is constant over the cross section.

The maximum \( \sigma_1 \) occurs at \( \psi = 0 \), the inner radius of the torus.

\[
\sigma_{1\text{max}} = \frac{pr}{2t} \left( \frac{2R - r}{R - r} \right)
\]

for yielding to occur, using the von Mises yield criterion:

\[
\sigma_y = \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2}
\]

substituting and arranging, the pressure to initiate yielding is given by:

\[
p_y = \frac{2\sigma_y t}{r} \left[ 3 + \frac{r}{R - r} + \left( \frac{r}{R - r} \right)^2 \right]^{-1/2}
\]
The minimum $\sigma_1$ occurs at $\psi = \pi$, the inner radius of the torus.

$$\sigma_{1\text{max}} = \frac{pr}{2t} \left( \frac{2R + r}{R + r} \right)$$

Similarly, for yielding to occur at the outer radius, using the von Mises yield criterion:

$$\sigma_y = \sqrt{\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2}$$

substituting and arranging, the pressure for full yielding is given by:

$$p_y = \frac{2\sigma_y t}{r} \left[ 3 - \frac{r}{R + r} + \left( \frac{r}{R + r} \right)^2 \right]^{-1/2}$$